ORTHOGONAL BASES OF BRAUER SYMMETRY CLASSES OF TENSORS FOR THE DIHEDRAL GROUP

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Abstract. Necessary and sufficient conditions are given for the existence of an orthogonal basis consisting of standard (decomposable) symmetrized tensors for the class of tensors symmetrized using a Brauer character of the dihedral group.

0. Introduction

Since the appearance in [WG] of an example of a (higher) symmetry class of tensors that possesses an orthogonal basis of standard (decomposable) symmetrized tensors ("o-basis" for short) there have been several papers devoted to the question of when such bases exist (e.g., [BPR, DP, DT, H1, H2, HT, SAJ]). In each of these papers the symmetrization of the tensors is done using an (ordinary) irreducible character of the given group. It makes sense however to symmetrize using an irreducible *Brauer* character of the group instead.

In this paper, we consider the above question in the case of these "Brauer-symmetry classes" of tensors. We give necessary and sufficient conditions for the existence of an o-basis in the case where the group is the dihedral group.

It is our hope that, just as the study of Brauer characters has revealed so much about ordinary characters, so too will the study of Brauer symmetrized tensors shed light on the structure of ordinary symmetrized tensors.

1. Notation and background

Fix positive integers m and n and set $\Gamma_{m,n} = \{\gamma \in \mathbf{Z}^m \,|\, 1 \leq \gamma_i \leq n\}$. Fix a subgroup G of the symmetric group S_m. A right action of G on the set $\Gamma_{m,n}$ is given by $\gamma\sigma = (\gamma_{\sigma(1)}, \dots, \gamma_{\sigma(m)})$ $(\gamma \in \Gamma_{m,n}, \sigma \in G)$.

Let V be a complex inner product space of finite dimension n and let $\{e_i \,|\, 1 \leq i \leq n\}$ be an orthonormal basis for V. The inner product on V induces an inner product on $V^{\otimes m}$ (the mth tensor power of V) and, with respect to this inner product, the set $\{e_\gamma \,|\, \gamma \in \Gamma_{m,n}\}$ is an orthonormal basis for V, where $e_\gamma = e_{\gamma_1} \otimes \cdots \otimes e_{\gamma_m}$.

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The space $V^{\otimes m}$ is a left $\mathbf{C}G$-module with action given by $\sigma e_\gamma = e_{\gamma\sigma^{-1}}$ ($\sigma \in G, \gamma \in \Gamma_{m,n}$), extended linearly. The inner product on $V^{\otimes m}$ is G-invariant, which is to say $(\sigma v, \sigma w) = (v, w)$ for all $\sigma \in G$ and all $v, w \in V^{\otimes m}$.

Let p be a fixed prime number. An element of G is *p-regular* if its order is not divisible by p. Denote by $\hat{G}$ the set of all p-regular elements of G. Let $\mathrm{IBr}(G)$ denote the set of irreducible Brauer characters of G. (A Brauer character is a certain function from $\hat{G}$ to $\mathbf{C}$ associated with an FG-module where F is a suitably chosen field of characteristic p. The Brauer character is irreducible if the associated module is simple. For the theory of Brauer characters, see [Is, Se, CR, Fe].) A conjugacy class of G consisting of p-regular elements is a *p'-conjugacy class*. The number of irreducible Brauer characters of G equals the number of p'-conjugacy classes of G.

Let $\mathrm{Irr}(G)$ denote the set of irreducible characters of G. (Unless preceded by the word "Brauer," the word "character" always refers to an ordinary character.) If the order of G is not divisible by p, then $\hat{G} = G$ and $\mathrm{IBr}(G) = \mathrm{Irr}(G)$.

Let S be a subset of G and let $\varphi : S \to \mathbf{C}$ be a fixed function. Statements below involving φ will hold in particular if φ is a character of G (in which case $S = G$) and also if φ is a Brauer character of G (in which case $S = \hat{G}$).

The *symmetrizer* corresponding to φ is the element s_φ of the group algebra $\mathbf{C}G$ given by

$$s_\varphi = \frac{\varphi(e)}{|S|} \sum_{\sigma \in S} \varphi(\sigma)\sigma.$$

Corresponding to φ and $\gamma \in \Gamma_{m,n}$ is the *standard* (or *decomposable*) *symmetrized tensor*

$$e_\gamma^\varphi = s_\varphi e_\gamma = \frac{\varphi(e)}{|S|} \sum_{\sigma \in S} \varphi(\sigma) e_{\gamma\sigma^{-1}}. \tag{1.0.1}$$

The *symmetry class of tensors* associated with φ is

$$V_\varphi = s_\varphi V^{\otimes n} = \langle e_\gamma^\varphi \mid \gamma \in \Gamma_{m,n} \rangle.$$

If φ is a Brauer character, we refer to this as the *Brauer symmetry class of tensors* associated with φ.

The *orbital subspace* of V_φ corresponding to $\gamma \in \Gamma_{m,n}$ is

$$V_\gamma^\varphi = \langle e_{\gamma\sigma}^\varphi \mid \sigma \in G \rangle.$$

An *o-basis* of a subspace W of V_φ is an orthogonal basis of W of the form $\{e_{\gamma_1}^\varphi, e_{\gamma_2}^\varphi, \ldots, e_{\gamma_t}^\varphi\}$ for some $\gamma_i \in \Gamma_{m,n}$. By convention, the empty set is regarded as an o-basis of the zero subspace of V_φ.

Let $\Delta = \Delta_G$ be a set of representatives of the orbits of $\Gamma_{m,n}$ under the action of G.

The following result is well known in the case where φ is an irreducible character of G [Me, Theorem 6.31 and proof], but we provide a proof to show that it holds for our arbitrary function $\varphi : S \to \mathbf{C}$ as well.

1.1 Theorem. *We have*

$$V_\varphi = \dot{\sum_{\gamma\in\Delta}} V_\gamma^\varphi \quad (\textit{orthogonal direct sum}).$$

In particular, V_φ *has an o-basis if and only if* V_γ^φ *has an o-basis for each* $\gamma \in \Delta$*.*

Proof. Let $\beta \in \Gamma_{m,n}$. Then $\beta = \gamma\sigma$ for some $\gamma \in \Delta$ and $\sigma \in G$, so that $e_\beta^\varphi = e_{\gamma\sigma}^\varphi \in V_\gamma^\varphi$. This shows that V_φ is contained in (and hence equals) the indicated sum.

The sets $E_\gamma = \{e_{\gamma\sigma} \mid \sigma \in G\}$, $\gamma \in \Delta$, are pairwise disjoint subsets of the orthogonal set $\{e_\beta \mid \beta \in \Gamma_{m,n}\}$ and are therefore pairwise orthogonal. For each $\gamma \in \Delta$ the subspace V_γ^φ is contained in the span of E_γ, so the indicated sum is an orthogonal direct sum.

Assume that V_φ has an o-basis B. By the first paragraph, B is the union of the sets $B_\gamma = B \cap V_\gamma^\varphi$, $\gamma \in \Delta$, and these sets are pairwise disjoint by the second paragraph, so B_γ is an o-basis for V_γ^φ for each $\gamma \in \Delta$.

Finally, if V_γ^φ has an o-basis for each $\gamma \in \Delta$, then the union of these bases is an o-basis for V_φ. □

1.2 Theorem. *For every* $\gamma \in \Gamma_{m,n}$ *and* $\sigma \in G$*,*

$$(e_{\gamma\sigma}^\varphi, e_\gamma^\varphi) = \frac{\varphi(e)^2}{|S|^2} \sum_{\mu\in S} \sum_{\tau\in\sigma\mu^{-1}S\cap G_\gamma} \varphi(\mu)\varphi(\tau^{-1}\sigma\mu^{-1}).$$

Proof. Let $\gamma \in \Gamma_{m,n}$ and $\sigma \in G$. Using the definition of the standard symmetrized tensor and sesquilinearity of the inner product, we have

$$(e_{\gamma\sigma}^\varphi, e_\gamma^\varphi) = \frac{\varphi(e)^2}{|S|^2} \sum_{\mu\in S} \sum_{\rho\in S} \varphi(\mu)\overline{\varphi(\rho)}(e_{\gamma\sigma\mu^{-1}}, e_{\gamma\rho^{-1}}).$$

For $\mu, \rho \in S$, the G-invariance of the inner product gives

$$(e_{\gamma\sigma\mu^{-1}}, e_{\gamma\rho^{-1}}) = (\rho^{-1}e_{\gamma\sigma\mu^{-1}}, \rho^{-1}e_{\gamma\rho^{-1}}) = (e_{\gamma\sigma\mu^{-1}\rho}, e_\gamma)$$

and this last expression is 1 or 0 according as $\sigma\mu^{-1}\rho$ is or is not in G_γ. Therefore,

$$\begin{aligned}(e_{\gamma\sigma}^\varphi, e_\gamma^\varphi) &= \frac{\varphi(e)^2}{|S|^2} \sum_{\mu\in S} \sum_{\substack{\rho\in S\\ \sigma\mu^{-1}\rho\in G_\gamma}} \varphi(\mu)\varphi(\rho^{-1}) \\ &= \frac{\varphi(e)^2}{|S|^2} \sum_{\mu\in S} \sum_{\tau\in\sigma\mu^{-1}S\cap G_\gamma} \varphi(\mu)\varphi(\tau^{-1}\sigma\mu^{-1}),\end{aligned}$$

where we have used the substitution $\rho = \mu\sigma^{-1}\tau$ in the last step. □

1.3 Lemma. *Assume that S is closed under conjugation by elements of G and that φ is constant on the conjugacy classes of G. For each $\gamma \in \Gamma_{m,n}$ and $\sigma \in G$, we have $\sigma e_\gamma^\varphi = e_{\gamma\sigma^{-1}}^\varphi$.*

Proof. Let $\gamma \in \Gamma_{m,n}$ and $\sigma \in G$. Then

$$\sigma e_\gamma^\varphi = \frac{\varphi(e)}{|S|} \sum_{\tau \in S} \varphi(\tau) e_{\gamma\tau^{-1}\sigma^{-1}}.$$

For $\tau \in S$, we have $\tau^{-1}\sigma^{-1} = \sigma^{-1}\mu^{-1}$, where $\mu = \sigma\tau\sigma^{-1}$, so

$$\sigma e_\gamma^\varphi = \frac{\varphi(e)}{|S|} \sum_{\mu \in S} \varphi(\sigma^{-1}\mu\sigma) e_{\gamma\sigma^{-1}\mu^{-1}} = \frac{\varphi(e)}{|S|} \sum_{\mu \in S} \varphi(\mu) e_{\gamma\sigma^{-1}\mu^{-1}} = e_{\gamma\sigma^{-1}}^\varphi,$$

where the first equality uses the assumption that S is closed under conjugation and the second equality uses the assumption that φ is constant on conjugacy classes. ☐

The preceding lemma applies in particular if φ is either a character or a Brauer character of G.

For the remainder of the section we review some standard results we will need from the theory of (ordinary) symmetrized tensors (some suitably generalized for our purposes).

1.4 Theorem. *We have*

$$V^{\otimes m} = \dot{\sum_{\chi \in \mathrm{Irr}(G)}} V_\chi \quad (\textit{orthogonal direct sum}).$$

Proof. See [Me, Corollary 6.6]. ☐

The preceding theorem does not hold with $\mathrm{Irr}(G)$ replaced by $\mathrm{IBr}(G)$ as Example 2.5 below shows.

The next result is well known, but we provide a proof that utilizes Theorem 1.2.

1.5 Theorem. *Let χ be an irreducible character of G. For every $\gamma \in \Gamma_{m,n}$ and $\sigma \in G$,*

$$(e_{\gamma\sigma}^\chi, e_\gamma^\chi) = \frac{\chi(e)}{|G|} \sum_{\tau \in G_\gamma} \chi(\tau\sigma).$$

Proof. Let $\gamma \in \Gamma_{m,n}$ and $\sigma \in G$. Letting $S = G$ in Theorem 1.2 and switching the order of the sums, we get

$$(e_{\gamma\sigma}^\chi, e_\gamma^\chi) = \frac{\chi(e)}{|G|} \sum_{\tau \in G_\gamma} \left[\frac{\chi(e)}{|G|} \sum_{\mu \in G} \chi(\mu)\chi(\tau^{-1}\sigma\mu^{-1}) \right].$$

The expression in brackets equals $\chi(\tau^{-1}\sigma)$ by the generalized orthogonality relation [Is, (2.13)], so replacing τ^{-1} by τ yields the desired formula. ☐

Fix a character ψ of G. We have

$$\psi = \sum_{i=1}^{t} a_i \chi_i, \tag{1.5.1}$$

with the χ_i distinct irreducible characters of G and the a_i positive integers. The characters χ_i $(1 \leq i \leq t)$ are the *irreducible constituents* of ψ.

The following theorem is a slight generalization of a theorem of Freese, which says that for every $\chi \in \mathrm{Irr}(G)$ and $\gamma \in \Gamma_{m,n}$

$$\dim V_\gamma^\chi = \chi(e)(\chi, 1)_{G_\gamma} \tag{1.5.2}$$

[Fe, p. 339].

1.6 Theorem. *For every $\gamma \in \Gamma_{m,n}$,*

$$\dim V_\gamma^\psi = \sum_{i=1}^{t} \chi_i(e)(\chi_i, 1)_{G_\gamma}.$$

where $\chi_1, \ldots, \chi_t$ are the (distinct) irreducible constituents of ψ.

Proof. Let $\gamma \in \Gamma_{m,n}$. Writing $\psi = \sum_i a_i \chi_i$ as in (1.5.1), we have $s_\psi = \sum_i c_i s_{\chi_i}$, where $c_i = a_i \psi(e)/\chi_i(e) \neq 0$. We claim that

$$V_\gamma^\psi = \sum_i V_\gamma^{\chi_i}. \tag{1.6.1}$$

We have

$$V_\gamma^\psi = s_\psi V_\gamma \subseteq \sum_i c_i s_{\chi_i} V_\gamma = \sum_i V_\gamma^{\chi_i},$$

where $V_\gamma = \langle e_{\gamma\sigma} \,|\, \sigma \in G \rangle$. Therefore, it remains to show the other inclusion. Let $v \in \sum_i V_\gamma^{\chi_i}$. Then $v = \sum_i s_{\chi_i} v_i$ with $v_i \in V_\gamma$. Using the fact that the symmetrizers s_χ, $\chi \in \mathrm{Irr}(G)$, are pairwise orthogonal idempotents [Me, Theorems 6.3 and 6.5] we get

$$v = \sum_i s_{\chi_i} v_i = \left(\sum_i c_i s_{\chi_i} \right) \left(\sum_j c_j^{-1} s_{\chi_j} v_j \right) \in s_\psi V_\gamma = V_\gamma^\psi,$$

and the claim is established.

Since the spaces V_{χ_i} are pairwise orthogonal (Theorem 1.4), the sum in 1.6.1 is direct. The theorem now follows from Freese's theorem (1.5.2). □

Define

$$\bar{\Delta} = \bar{\Delta}_G^\psi = \{\gamma \in \Delta \,|\, (\psi, 1)_{G_\gamma} \neq 0\}.$$

1.7 Proposition. *Let $\gamma \in \Delta$. The following are equivalent:*

(i) $\gamma \in \bar{\Delta}$,

(ii) $V_\gamma^\psi \neq \{0\}$,

(iii) $e_{\gamma\sigma}^\psi \neq 0$ *for every* $\sigma \in G$.

Proof. If we write $\psi = \sum_i a_i \chi_i$ as in (1.5.1) with each a_i a positive integer, we have $(\psi, 1)_{G_\gamma} = \sum_i a_i(\chi_i, 1)_{G_\gamma}$, so (i) and (ii) are equivalent by Lemma 1.6. Next, by Lemma 1.3, $e^{\psi}_{\gamma\sigma} \neq 0$ for every $\sigma \in G$ if and only if $e^{\psi}_{\gamma\sigma} \neq 0$ for *some* $\sigma \in G$. Since V^{ψ}_{γ} is spanned by the vectors $e^{\psi}_{\gamma\sigma}$ with $\sigma \in G$, it follows that (ii) and (iii) are equivalent. $\square$

2. The dihedral group

Assume that $m \geq 3$ and define $r, s \in S_m$ by

$$r = \begin{pmatrix} 1 & 2 & 3 & \cdots & m-1 & m \\ 2 & 3 & 4 & \cdots & m & 1 \end{pmatrix}, \quad s = \begin{pmatrix} 1 & 2 & 3 & \cdots & m-1 & m \\ 1 & m & m-1 & \cdots & 3 & 2 \end{pmatrix}.$$

Then $G = D_m = \langle r, s \rangle$ is the *dihedral group* of degree m. The elements r and s satisfy the relations $r^m = 1$, $s^2 = 1$, and $sr^k s = r^{-k}$ for all k. It follows that $G = \{r^k, sr^k \mid 0 \leq k < m\}$. The order of G is $2m$.

The ordinary irreducible characters of G are [Se, pp. 37–38]

	r^k	sr^k	
ψ_0	1	1	
ψ_1	1	-1	
ψ_2	$(-1)^k$	$(-1)^k$	(m even)
ψ_3	$(-1)^k$	$(-1)^{k+1}$	(m even)
χ_h	$2\cos\frac{2\pi kh}{m}$	0	$(1 \leq h < m/2)$

Let C denote the cyclic subgroup $\langle r \rangle$ of G and fix $1 \leq h < m/2$. The character χ_h is induced from the character λ_h of C given by $\lambda_h(r^k) = \omega^{hk}$, where $\omega = e^{2\pi i/m}$ $(i = \sqrt{-1})$, a primitive mth root of unity. Moreover, $\chi_h|_C = \lambda_h + \bar{\lambda}_h$, where $\bar{\lambda}_h$ denotes the conjugate character of λ_h. In particular, for each k we have

$$\chi_h(r^k) = \lambda_h(r^k) + \bar{\lambda}_h(r^k) = \omega^{hk} + \omega^{-hk}.$$

Write $m = p^t l$ with l an integer not divisible by p (where p is our fixed prime number). We have

$$\hat{G} = \begin{cases} \{r^{jp^t}, sr^k \mid 0 \leq j < l,\ 0 \leq k < m\}, & p \neq 2, \\ \{r^{jp^t} \mid 0 \leq j < l\}, & p = 2. \end{cases}$$

so the p'-conjugacy classes of G are

$$\begin{array}{ll} \{r^{jp^t}, r^{-jp^t}\},\ 0 \leq j \leq l/2, & \\ \qquad \{sr^{2k} \mid 0 \leq k < m/2\},\ \{sr^{2k+1} \mid 0 \leq k < m/2\}, & l \text{ even}, p \neq 2, \\ \{r^{jp^t}, r^{-jp^t}\},\ 0 \leq j \leq (l-1)/2,\ \{sr^k \mid 0 \leq k < m\}, & l \text{ odd}, p \neq 2, \\ \{r^{jp^t}, r^{-jp^t}\},\ 0 \leq j < l/2, & p = 2. \end{array}$$

Define

$$\varepsilon = \begin{cases} 4, & l \text{ even}, p \neq 2, \\ 2, & l \text{ odd}, p \neq 2, \\ 1, & p = 2. \end{cases}$$

For each j and h put

$$\hat{\psi}_j = \psi_j|_{\hat{G}}, \qquad \hat{\chi}_h = \chi_h|_{\hat{G}}.$$

2.1 Proposition. *The complete list of irreducible Brauer characters of G is $\hat{\psi}_j$ $(0 \leq j < \epsilon)$, $\hat{\chi}_h$ $(1 \leq h < l/2)$.*

Proof. The restriction of a character of G to $\hat{G}$ is a Brauer character, and the number of the $\hat{\psi}_j$ and $\hat{\chi}_h$ with indices ranging as indicated is the same as the number of p'-conjugacy classes of G, so it suffices to show that these Brauer characters are all irreducible and distinct.

Each $\hat{\psi}_j$ has degree one and is therefore irreducible. Also, the $\hat{\psi}_j$ with $0 \leq j < \varepsilon$ are distinct. This can be seen in the case $p \neq 2$ by looking at the column labeled sr^k in the character table; if $p = 2$, then $\varepsilon = 1$ so the issue of distinctness does not arise.

Fix h with $1 \leq h < l/2$ and suppose that $\hat{\chi}_h$ is not irreducible. Then $\hat{\chi}_h$ is a sum of two Brauer characters, each of degree one. Since G is solvable, the Fong-Swan theorem says that every irreducible Brauer character of G is the restriction to $\hat{G}$ of an ordinary irreducible character of G [Fe, Theorem 2.1, p. 419]. Thus, $\hat{\chi}_h = \hat{\psi}_j + \hat{\psi}_k$ for some j and k. It follows from our assumption $1 \leq h < l/2$ that $l \geq 3$, so $r^{2p^t} \in \hat{G}$ and

$$\hat{\chi}_h(r^{2p^t}) = \hat{\psi}_j(r^{2p^t}) + \hat{\psi}_k(r^{2p^t}) = 2.$$

On the other hand,

$$\hat{\chi}_h(r^{2p^t}) = \chi_h(r^{2p^t}) = 2\cos\frac{4\pi p^t h}{m} = 2\cos\frac{4\pi h}{l},$$

and, since $1 \leq h < l/2$, the argument of the cosine is strictly between 0 and 2π, so this last expression does not equal 2, a contradiction. We conclude that $\hat{\chi}_h$ is irreducible.

Finally, suppose that $\hat{\chi}_h = \hat{\chi}_k$ with $1 \leq h \leq k < l/2$. Evaluating both sides at r^{p^t} yields

$$2\cos\frac{2p^t h\pi}{m} = 2\cos\frac{2p^t k\pi}{m}.$$

Thus the arguments of the cosines differ by an integer multiple of 2π, which leads to the equation $k - h = jl$ for some $j \in \mathbf{Z}$. But $0 \leq k - h < l/2$, so $j = 0$ and $k = h$. This shows that the $\hat{\chi}_h$ are distinct and finishes the proof. $\square$

2.2 Theorem. *Let $G = D_m$, let $0 \leq j < \varepsilon$, and put $\varphi = \hat{\psi}_j$. The space V_φ has an o-basis if and only if at least one of the following holds:*

(i) *$\dim V = 1$,*
(ii) *$p = 2$,*
(iii) *m is not divisible by p.*

Proof. If $\dim V = 1$, then $V_\varphi = \langle e_\gamma^\varphi \rangle$, where $\gamma = (1, 1, \dots, 1)$, so V_φ has o-basis $\{e_\gamma^\varphi\}$ or $\emptyset$ according as $\dim V_\varphi$ is 1 or 0.

Assume that $p = 2$. Then φ is an (ordinary) irreducible character of the group $\hat{G} = \langle r^{p^t} \rangle$. In particular, each orbital subspace V_γ^φ, $\gamma \in \Delta_{\hat{G}}$, has dimension at most one by Freese's Theorem (1.5.2), and so has an o-basis. By Theorem 1.1, V_φ has an o-basis.

Assume that m is not divisible by p. Then $\hat{G} = G$ and φ equals the ordinary character ψ_i. Since ψ_i has degree one, each orbital subspace has dimension at most one and so V_φ has an o-basis by the argument of the preceding paragraph.

Now assume that $\dim V \neq 1$, $p \neq 2$, and m is divisible by p. We will exhibit $\gamma \in \Delta$ for which V_γ^φ does not have an o-basis, and then it will follow from Theorem 1.1 that V_φ does not have an o-basis.

First assume that $m \geq 6$. Let $\gamma = (1, 2, 1, 1, \dots, 1, 2, 2)$, which may be assumed to lie in Δ, and note that $G_\gamma = \{e\}$. Let $\sigma \in G$. In the formula of Theorem 1.2 with $S = \hat{G}$ we have $\varphi(\mu)\varphi(\tau^{-1}\sigma\mu^{-1}) = \varphi(\tau^{-1}\sigma)$ since $\varphi = \psi_i|_{\hat{G}}$ and ψ_i is linear and hence a homomorphism, so

$$(e_{\gamma\sigma}^\varphi, e_\gamma^\varphi) = \frac{\varphi(e)^2}{|\hat{G}|^2} \psi_i(\sigma)|X|,$$

where $X = \{\mu \in \hat{G} \,|\, e \in \sigma\mu^{-1}\hat{G}\}$. Since $e \in \sigma\mu^{-1}\hat{G}$ if and only if $\sigma \in \hat{G}^{-1}\mu = \hat{G}\mu$, we have $X = \{\mu \in \hat{G} \,|\, \sigma \in \hat{G}\mu\}$. Now $\hat{G} = \{r^{jp^t}, sr^k \,|\, 0 \leq j < l,\ 0 \leq k < m\}$, and for arbitrary $0 \leq k < m$ we have $r^k = sr^0 sr^k \in \hat{G}^2$ and $sr^k = r^0 sr^k \in \hat{G}^2$, so $G \subseteq \hat{G}^2$. Therefore, X is nonempty and the expression above is nonzero. We have shown that $(e_{\gamma\sigma}^\varphi, e_\gamma^\varphi) \neq 0$ for all $\sigma \in G$.

Since $\{e_\delta \,|\, \delta \in \Gamma_{m,n}\}$ is a basis for $V^{\otimes n}$, we have $e_\gamma^\varphi = \sum_\delta a_\delta e_\delta$ and $e_{\gamma r}^\varphi = \sum_\delta b_\delta e_\delta$ for unique $a_\delta, b_\delta \in \mathbf{C}$. Now $\hat{G}^{-1} = \hat{G}$, so Equation 1.0.1 gives

$$e_\gamma^\varphi = \frac{\varphi(e)}{|\hat{G}|} \sum_{\sigma \in \hat{G}} \varphi(\sigma^{-1}) e_{\gamma\sigma}.$$

Since p divides m (implying $t \neq 0$) and $p \geq 3$ it follows that the first entry of γr^{jp^t} is 1 for each $0 \leq j < l$. Also, for each $0 \leq k < m$, γsr^k is a cyclic permutation of $\gamma s = (1, 2, 2, 1, 1, \dots, 1, 2)$. Therefore, $\gamma\sigma \neq (2, 1, 2, 1, 1, \dots, 1, 2) = \gamma r$ for all $\sigma \in \hat{G}$, implying $a_{\gamma r} = 0$. On the other hand, $G_{\gamma r} = r^{-1} G_\gamma r = \{e\}$, so for $\sigma \in \hat{G}$, $(\gamma r)\sigma = \gamma r$ if and only if $\sigma = e$, implying $b_{\gamma r} = |\hat{G}|^{-1}$. Therefore, e_γ^φ and $e_{\gamma r}^\varphi$ are linearly independent elements of V_γ^φ. In particular, $\dim V_\gamma^\varphi > 1$. By Lemma 1.3 and the G-invariance of the inner product, if V_γ^φ were to have an o-basis, then it would have an o-basis containing e_γ^φ, but in view of the preceding paragraph, we see that this is not the case. We conclude that V_γ^φ does not have an o-basis.

The foregoing argument assumed that $m \geq 6$. Since m is divisible by p and $p \neq 2$, this leaves the two cases $(m, p) = (3, 3)$ and $(m, p) = (5, 5)$. First assume that $(m, p) = (3, 3)$ and let $\gamma = (1, 2, 2)$, which we may assume lies in Δ. Then $G_\gamma = \{e, s\}$ and the set of those standard symmetrized tensors contained in V_γ^φ is $\{e_{122}^\varphi, e_{212}^\varphi, e_{221}^\varphi\}$, where we are writing 122 for $(1, 2, 2)$, etc. Now,

$$\begin{aligned} e_{122}^\varphi &= \frac{1}{4}[(1 \pm 1)e_{122} + (\pm 1)e_{212} + (\pm 1)e_{221}] \\ e_{212}^\varphi &= \frac{1}{4}[(\pm 1)e_{122} + (1 \pm 1)e_{212} + (\pm 1)e_{221}] \\ e_{221}^\varphi &= \frac{1}{4}[(\pm 1)e_{122} + (\pm 1)e_{212} + (1 \pm 1)e_{221}], \end{aligned}$$

where we use the $(+)$ signs or the $(-)$ signs according as φ is ψ_1 or ψ_2. The matrix of coefficients on the right is nonsingular, so these three vectors are linearly independent. Therefore, $\dim V_\gamma^\varphi = 3$. On the other hand, these vectors are not pairwise orthogonal. We conclude that V_γ^φ does not have an o-basis. An almost identical argument leads to the same conclusion in the case $(m, p) = (5, 5)$ using $\gamma = (1, 2, 2, 2, 2)$.

In each case, we have found $\gamma \in \Delta$ for which V_γ^φ does not have an o-basis. As remarked earlier, it follows that V_φ does not have an o-basis either, and the proof is complete. □

The preceding theorem shows that, in contrast to the case with ordinary symmetrized tensors, it is possible for a symmetry class of tensors corresponding to a linear (i.e., degree one) Brauer character to fail to have an o-basis.

Recall that we are writing $m = p^t l$ with l an integer not divisible by p.

2.3 Theorem. *Let $G = D_m$, let $1 \leq h < l/2$, and put $\varphi = \hat{\chi}_h$. The vector space V_φ has an o-basis if and only if either $\dim V = 1$ or l' is divisible by 4, where $l' = l/\gcd(l, h)$.*

Proof. Denote by ψ the restriction of φ to $\hat{C} = C \cap \hat{G} = \langle r^{p^t} \rangle$ and denote by η the restriction of λ_h to $\hat{C}$, so that

$$\psi = \varphi|_{\hat{C}} = (\lambda_h + \bar{\lambda}_h)|_{\hat{C}} = \eta + \bar{\eta}.$$

Step 1: We have $\eta \neq \bar{\eta}$.

Suppose $\eta = \bar{\eta}$. Then

$$\omega^{hp^t} = \lambda_h(r^{p^t}) = \eta(r^{p^t}) = \bar{\eta}(r^{p^t}) = \bar{\lambda}_h(r^{p^t}) = \omega^{-hp^t},$$

implying $\omega^{2hp^t} = 1$ so that $2hp^t = km = kp^t l$ for some positive integer k. But this implies $0 < kl = 2h < l$, yielding the contradiction $0 < k < 1$. Therefore, $\eta \neq \bar{\eta}$ as claimed.

Step 2: If $\gamma \in \bar{\Delta}_{\hat{C}}^\psi$, then $(\eta, 1)_{\hat{C}_\gamma} = 1$ and $(\bar{\eta}, 1)_{\hat{C}_\gamma} = 1$.

Let $\gamma \in \bar{\Delta}^{\psi}_{\hat{C}}$. Since $\psi = \eta + \bar{\eta}$, we have

$$0 \neq (\psi, 1)_{\hat{C}_\gamma} = (\eta, 1)_{\hat{C}_\gamma} + (\bar{\eta}, 1)_{\hat{C}_\gamma}. \tag{2.3.1}$$

Each term on the right is a nonnegative integer, so at least one of the terms must be nonzero. But

$$(\bar{\eta}, 1)_{\hat{C}_\gamma} = (\bar{\eta}, \bar{1})_{\hat{C}_\gamma} = \overline{(\eta, 1)}_{\hat{C}_\gamma} = (\eta, 1)_{\hat{C}_\gamma},$$

so we conclude that both terms on the right of (2.3.1) are nonzero. Since η and $\bar{\eta}$ are linear both of these terms must be 1 as claimed.

Step 3: For $\gamma \in \bar{\Delta}^{\psi}_{\hat{C}}$ and $\sigma \in \hat{C}$ we have

$$(e^{\psi}_{\gamma\sigma}, e^{\psi}_{\gamma}) = c\big(\eta(\sigma) + \bar{\eta}(\sigma)\big),$$

where $c = 4|\hat{C}_\gamma|/|\hat{C}|$.

Let $\gamma \in \bar{\Delta}^{\psi}_{\hat{C}}$ and $\sigma \in \hat{C}$. We have $\psi = \eta + \bar{\eta}$, and η and $\bar{\eta}$ are distinct irreducible characters of $\hat{C}$ by Step 1, so due to the orthogonality of V_η and $V_{\bar{\eta}}$ (Theorem 1.4), we get

$$\begin{aligned}(e^{\psi}_{\gamma\sigma}, e^{\psi}_{\gamma}) &= 4(e^{\eta}_{\gamma\sigma} + e^{\bar{\eta}}_{\gamma\sigma}, e^{\eta}_{\gamma} + e^{\bar{\eta}}_{\gamma}) \\ &= 4(e^{\eta}_{\gamma\sigma}, e^{\eta}_{\gamma}) + 4(e^{\bar{\eta}}_{\gamma\sigma}, e^{\bar{\eta}}_{\gamma}). \end{aligned} \tag{2.3.2}$$

Using the fact that η has degree one and is therefore a homomorphism, we get from Theorem 1.5 that

$$\begin{aligned}(e^{\eta}_{\gamma\sigma}, e^{\eta}_{\gamma}) &= \frac{1}{|\hat{C}|}\sum_{\tau \in \hat{C}_\gamma} \eta(\tau\sigma) = \eta(\sigma)\frac{1}{|\hat{C}|}\sum_{\tau \in \hat{C}_\gamma} \eta(\tau) \\ &= \eta(\sigma)\frac{|\hat{C}_\gamma|}{|\hat{C}|}(\eta, 1)_{\hat{C}_\gamma} = \eta(\sigma)\frac{|\hat{C}_\gamma|}{|\hat{C}|},\end{aligned}$$

where the last equality uses Step 2. The same formula with η replaced by $\bar{\eta}$ holds, so the claim follows.

We are now ready to proceed with the proof of the theorem. If $p \neq 2$, then $\hat{G} = \hat{C} \cup \{sr^k \,|\, 0 \leq k < m\}$, and if $p = 2$, then $\hat{G} = \hat{C}$. In either case, the Brauer character φ vanishes off the subgroup $\hat{C}$, so $s_\varphi = cs_\psi$, where $c = |\hat{C}|/|\hat{G}|$. Therefore, $V_\varphi = V_\psi$ and $e^{\varphi}_{\gamma} = ce^{\psi}_{\gamma}$ for each $\gamma \in \Gamma_{m,n}$, so it suffices to prove the theorem with φ replaced by ψ.

Assume that V_ψ has an o-basis and $\dim V \neq 1$. Let $\gamma = (1, 2, 2, \dots, 2) \in \Gamma_{m,n}$. We may assume that $\gamma \in \Delta_{\hat{C}}$ and, since $\hat{C}_\gamma = \{e\}$, it follows that $\gamma \in \bar{\Delta}^{\psi}_{\hat{C}}$. We have $\psi = \eta + \bar{\eta}$ and, since η and $\bar{\eta}$ are distinct irreducible characters of $\hat{C}$ by Step 1, it follows from Theorem 1.6 and Step 2 that $\dim V^{\psi}_{\gamma} = 2$. Therefore, by our assumption and Theorem 1.1 (with $\hat{C}$ playing the role of S), V^{ψ}_{γ} has an o-basis consisting of two elements. Due to the $\hat{C}$-invariance of the inner product and Lemma 1.3, we may assume that V^{ψ}_{γ} has an o-basis of the form $\{e^\psi_\gamma, e^\psi_{\gamma\sigma}\}$ for some $\sigma \in \hat{C} = \langle r^{p^t} \rangle$. We have $\sigma = r^{p^t j}$ for some integer j, so Step 3 gives

$$0 = (e^\psi_{\gamma\sigma}, e^\psi_\gamma) = c(\omega^{hjp^t} + \omega^{-hjp^t}),$$

where $c = 4|\hat{C}_\gamma|/|\hat{C}|$, whence $\omega^{2hjp^t} = -1$. This implies that m is even and $2hjp^t = \frac{m}{2} + km$ for some integer k, which in turn implies that $4h'j = l'(1+2k)$, where $h' = h/\gcd(l,h)$ and $l' = l/\gcd(l,h)$. Therefore, l' is divisible by 4, as desired.

We now prove the converse. If $\dim V = 1$, then the argument in the proof of Theorem 2.2 shows that V_ψ has an o-basis.

Now assume that l' is divisible by 4 so that $l' = 4k$ for some integer k. In order to show that V_ψ has an o-basis, it suffices by Theorem 1.1 and Proposition 1.7 to show that V^ψ_γ has an o-basis for each $\gamma \in \bar{\Delta}^\psi_{\hat{C}}$, so fix such a γ.

We claim that $\{e^\psi_\gamma, e^\psi_{\gamma\sigma}\}$ is an o-basis for V^ψ_γ, where $\sigma = r^{p^t k}$. Arguing as above, we have $\dim V^\psi_\gamma = 2$, and the vectors e^ψ_γ and $e^\psi_{\gamma\sigma}$ are nonzero by Proposition 1.7, so it suffices to show that these vectors are orthogonal. Since $l' = l/\gcd(l,h)$ and $h' = h/\gcd(l,h)$ are relatively prime, we have $1 = al' + bh'$ for some integers a and b. Then $bh' = 1 - 4ak$, so h' is odd. Therefore, with $c = 4|\hat{C}_\gamma|/|\hat{C}|$, Step 3 gives

$$\begin{aligned} c^{-1}(e^\psi_{\gamma\sigma}, e^\psi_\gamma) &= \omega^{hp^t k} + \omega^{-hp^t k} = (\omega^{(m/4)})^{h'} + (\omega^{-(m/4)})^{h'} \\ &= i^{h'} + (-i)^{h'} = i^{h'} - i^{h'} = 0, \end{aligned}$$

where $i = \sqrt{-1}$. This completes the proof. □

The special case of the preceding theorem where p does not divide the order of G recovers the following result by Holmes and Tam.

2.4 Corollary [HT, Theorem 3.1]. *Let $G = D_m$, let $\chi = \chi_h$ with $1 \leq h < m/2$, and assume that $\dim V \geq 2$. The space V_χ has an o-basis if and only if $m \equiv 0 \mod 4h_2$, where $h = h_2 h_{2'}$ with h_2 a power of 2 and $h_{2'}$ odd.*

Proof. Choose a prime p that does not divide the order of G. Then $\chi = \varphi_h$ and $m = l$, so Theorem 2.3 says that V_χ has an o-basis if and only if m' is divisible by 4, where $m' = m/\gcd(m,h)$. Therefore, it remains to show that this last condition is equivalent to the one in the statement of the theorem.

Assume that $m \equiv 0 \mod 4h_2$. Then $m = 4h_2 k$ for some integer k. We have $\gcd(m,h) = h_2 g$, where $g = \gcd(m, h_{2'})$, so $m' = m/\gcd(m,h) = 4(k/g)$. Now g is an odd integer that divides $m = 4h_2 k$ and $4h_2$ is a power of 2, so g divides k. Therefore, k/g is an integer and so m' is divisible by 4.

Now assume that m' is divisible by 4 so that $m' = 4k$ for some integer k. Put $c = \gcd(m,h)$. Then $c = am + bh$ for some integers a and b, so $(1 - 4ak)c = bh_2 h_{2'}$. It follows that h_2 divides c and we can write $c = h_2 d$ for some integer d. Thus, $m = m'c = 4kh_2 d$, which says that $m \equiv 0 \mod 4h_2$. This completes the proof. □

2.5 *Example.* Let $G = D_m$ and assume that $\dim V \geq 3$ and $m, p = 3$. Let $\gamma = (1,2,3) \in \Gamma_{m,n}$ and put $\varphi = \hat{\psi}_1$ and $\varphi' = \hat{\psi}_2$. Then $\varphi, \varphi' \in \mathrm{IBr}(G)$ and $\varphi \neq \varphi'$, yet

$$\begin{aligned}(e^{\varphi}_{\gamma}, e^{\varphi'}_{\gamma}) &= \frac{1}{16}(e_{123} + e_{132} + e_{213} + e_{321}, e_{123} - e_{132} - e_{213} - e_{321}) \\ &= -1/8 \neq 0.\end{aligned}$$

so V_φ is not orthogonal to $V_{\varphi'}$ (cf. remark after Theorem 1.4).

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